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# Oscillation of Certain Functional Differential Equations

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**Abstract**—In this paper, we shall establish some new criteria for the oscillation of functional differential equations of the form

$$x^{(n)}(t) + (-1)^n F\left(t, x(g(t)), \frac{d}{dt}x(h(t))\right) = 0$$

via comparing it with some other functional differential equations of the same or lower order whose oscillatory behavior is known. 

**Keywords**—Oscillation, Functional differential equations, Comparison.

## 1. INTRODUCTION

Consider the functional differential equation

$$x^{(n)}(t) + (-1)^n F\left(t, x(g(t)), \frac{d}{dt}x(h(t))\right) = 0, \tag{E}$$

where $g, h : [t_0, \infty) \to \mathbb{R} = (-\infty, \infty)$, $t \geq t_0$, and $F : [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $h(t) \leq t$, $h'(t) > 0$ for $t \geq t_0$, $h(t) \to \infty$, and $g(t) \to \infty$ as $t \to \infty$. We shall assume the following.

(P) There exists a continuous function $q : [t_0, \infty) \to [0, \infty), q(t) \neq 0$ for all large $t$ and positive real numbers $c$ and $c^*$ such that

$$F(t, x, y) \text{ sgn } x \geqq (t)|x|^{c^*}|y|^c, \qquad \text{for } xy \neq 0 \quad \text{and} \quad t \geq t_0.$$

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In what follows, we shall consider only the nonconstant solutions of (E) which are defined for all large $t$. The oscillatory behavior is considered in the usual sense, i.e., a solution of (E) is called oscillatory if it has no last zero, otherwise, it is called nonoscillatory. Equation (E) is said to be oscillatory if all of its nonconstant solutions are oscillatory.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of $n^{\text{th}}$ order functional differential equations with deviating arguments are oscillatory has been studied by a number of researchers. A large portion of these results have been for the equations of the form

$$x^{(n)}(t) + \delta F(t, x(g(t))) = 0, \tag{E$_1$}$$

where $\delta = \pm 1$ and the function $F$ satisfies a condition of type (P). For typical results for equation ($E_1$), we refer the readers to the papers [1–8] and the references therein. Results on the oscillatory behavior of solutions of ($E_1$) have been recently [3,9–11] extended to equations of the form

$$x^{(n)}(t) + (-1)^n p(t) x^{(n-1)}(t+h) + F(t, x(g(t))) = 0, \tag{E$_2$}$$

where $h = 0$ if $n$ is even and $h > 0$ if $n$ is odd, and to equations of the type (E) when $n$ is even and the deviating arguments are some constants [12].

In this paper, we shall offer some criteria for the oscillation of (E) via comparing it with some functional differential equations of the same or lower order whose oscillatory behavior is known. The results presented in Section 3 are concerned with the oscillation of (E) when $n$ is even. The obtained results extend and improve results established in [12]. Section 4 deals with the oscillation of (E) when $n$ is odd. These results appear to be new in the literature. To dwell upon the importance of our results, two examples are also illustrated.

## 2. PRELIMINARY RESULTS

We shall need the following.

LEMMA 1. *(See [5].) Let $u$ be a positive and $n$-times differentiable function on $[t_0, \infty)$. If $u^{(n)}$ is of constant sign and not identically zero on any interval $[t*, \infty)$ for some $t^* \geq t_0$, then there exist a $t_u \geq t_0$ and an integer $m$, $0 \leq m \leq n$ with $n + m$ even for $u^{(n)}$ nonnegative or $n + m$ odd for $u^{(n)}$ nonpositive and such that for $t \geq t_u$,*

$$u^{(k)}(t) > 0, \quad 0 \leq k \leq m \qquad \text{and} \qquad (-1)^{m+k} u^{(k)}(t) > 0, \quad m \leq k \leq n.$$

LEMMA 2. *(See [13].) Let $u$ be as in Lemma 1. In addition, let $\lim_{t\to\infty} u(t) \neq 0$ and $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for every $t \geq t_u$, then for every $b$, $0 < b < 1$, the following hold:*

$$u(t) \geq \frac{b}{(n-1)!} t^{n-1} u^{(n-1)}(t), \qquad \text{for all large } t.$$

## 3. OSCILLATION FOR EVEN ORDER EQUATIONS

THEOREM 1. *Let in addition to condition (P), $h(t) \leq g(t) \leq t$ for $t \geq t_0$. Further, let for every $a_i, 0 < a_i < 1, i = 1, 2$, the equations*

$$y'(t) + \left[\frac{a_1}{(n-1)^{c^*}((n-2)!)^m}\right] (h(t))^{(n-2)m} H(t) q(t) |y(h(t))|^m \operatorname{sgn} y(h(t)) = 0 \tag{1}$$

*and*

$$z'(t) + \left[\frac{a_2}{(2^{n-2}(n-2)!)^m}\right] (t - h(t))^{(n-2)m} H(t) q(t) \left| z\left(\frac{t + h(t)}{2}\right)\right|^m \operatorname{sgn} z\left(\frac{t + h(t)}{2}\right) = 0, \tag{2}$$

*where $c + c^* = m \leq 1$ and $H(t) = (h(t))^{c^*}(h'(t))^c$ are oscillatory, then equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 1, there exists a $t_1 \geq t_0$ such that $x^{(n-1)}(t) > 0$ and $x'(t) > 0$ for $t \geq t_1$. We distinguish the following two cases:

(I) $x^{(n)}(t) \leq 0$, $x^{(n-1)}(t) > 0, \dots, x''(t) > 0$ and $x'(t) > 0$ for $t \geq t_1$,
(II) $x^{(n)}(t) \leq 0$, $x^{(n-1)}(t) > 0, \dots, x''(t) < 0$ and $x'(t) > 0$ for $t \geq t_1$.

Assume (I) holds. By Lemma 2, there exist $t_2 \geq t_1$ and $b_i$, $0 < b_i < 1$, $i = 1, 2$, such that for $t \geq t_2$

$$x(g(t)) \geq x(h(t)) \geq \frac{b_1}{(n-1)!}(h(t))^{n-1}x^{(n-1)}(h(t)) \tag{3}$$

and

$$\frac{d}{dt}x(h(t)) = x'(h(t))h'(t) \geq \frac{b_2}{(n-2)!}(h(t))^{n-2}h'(t)x^{(n-1)}(h(t)). \tag{4}$$

Using (P), (3), and (4) in equation (E), we get

$$x^{(n)}(t) + \left(\frac{b_1}{(n-1)!}\right)^{c^*}\left(\frac{b_2}{(n-2)!}\right)^{c}(h(t))^{(n-2)m}H(t)q(t)\left(x^{(n-1)}(h(t))\right)^m \leq 0, \qquad t \geq t_2.$$

Setting $w(t) = x^{(n-1)}(t)$, $t \geq t_2$, we obtain

$$w'(t) + \left[\frac{b_1^{c^*}b_2^{c}}{((n-1)!)^{c^*}((n-2)!)^{c}}\right](h(t))^{(n-2)m}H(t)q(t)(w(h(t)))^m \leq 0, \qquad t \geq t_2. \tag{5}$$

Integrating (5) from $t$ to $T \geq t \geq t_2$ and letting $T \to \infty$, we find

$$w(t) \geq \left[\frac{b_1^{c^*}b_2^{c}}{(n-1)^{c^*}((n-2)!)^{m}}\right]\int_t^{\infty}(h(s))^{(n-2)m}H(s)q(s)(w(h(s)))^m\,ds.$$

The function $w(t) = x^{(n-1)}(t)$ is clearly strictly decreasing for $t \geq t_2$. Hence, by Theorem 1, in [17] there exists a positive solution $y(t)$ of (1) with $y(t) \to 0$ as $t \to \infty$. But, this contradicts the assumptions of our theorem.

Assume (II) holds. By Lemma 2, there exists a $T_1 \geq t_1$ and a constant $b, 0 < b < 1$ such that

$$x(g(t)) \geq x(h(t)) \geq bh(t)x'(h(t)), \qquad \text{for } t \geq T_1. \tag{6}$$

Using (P) and (6) in (E) and setting $v(t) = x'(t)$ for $t \geq T_1$, we get

$$v^{(n-1)}(t) + b^{c^*}H(t)q(t)(v(h(t)))^m \leq 0, \qquad \text{for } t \geq T_1. \tag{7}$$

It is clear that the function $v(t)$ satisfies

$$(-1)^i v^{(i)}(t) > 0, \qquad 0 \leq i \leq n-1, \quad \text{and} \quad \geq T_1. \tag{8}$$

Now by Lemma 1, in [8] there exists a $T \geq T_1$ such that

$$v(h(t)) \geq \left[\frac{(t-h(t))^{n-2}}{2^{n-2}(n-2)!}\right]v^{(n-2)}\left(\frac{t+h(t)}{2}\right), \qquad \text{for } T \leq h(t) \leq \frac{t+h(t)}{2}. \tag{9}$$

Thus, (7) takes the form

$$u'(t) + \left[\frac{b^{c^*}}{(2^{n-2}(n-2)!)^m}\right](t-h(t))^{(n-2)m}H(t)q(t)\left(u\left(\frac{t+h(t)}{2}\right)\right)^m \leq 0, \qquad t \geq T, \tag{10}$$

where $u(t) = v^{(n-2)}(t)$, $t \geq T$. The rest of the proof is similar to that of Case (I). ∎

Now we apply the results of [14] and [15] to Theorem 1 and obtain the following.

COROLLARY 1. *Let Condition (P) hold and $h(t) \le g(t) \le t$ for $t \ge t_0$. If $m = c + c^* = 1$, then*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} (h(s))^{n-2} H(s) q(s)\, ds > \frac{(n-1)^{c^*}(n-2)!}{e} \tag{11}$$

*and*

$$\liminf_{t\to\infty} \int_{(t+h(t))/2}^{t} (s-h(s))^{n-2} H(s) q(s)\, ds > \frac{2^{n-2}(n-2)!}{e}, \tag{12}$$

*and if $m = c + c^* < 1$, then*

$$\int^{\infty} (h(s))^{(n-2)m} H(s) q(s)\, ds = \infty \tag{13}$$

*and*

$$\int^{\infty} (s-h(s))^{(n-2)m} H(s) q(s)\, ds = \infty \tag{14}$$

*imply that equation (E) is oscillatory.*

In Theorem 1, if we let $B = \min\{a_1, a_2\}$, and

$$Q(t) = \min\left\{ \frac{(h(t))^{(n-2)m}}{(n-1)^{c^*}}, \left(\frac{t-h(t)}{2}\right)^{(n-2)m} \right\},$$

then we have the following oscillation criterion.

THEOREM 1′. *Let Condition (P) hold and $h(t) \le g(t) \le t$ for $t \ge t_0$. If for every $B$, $0 < B < 1$, equation*

$$w'(t) + \left[\frac{B}{((n-2)!)^m}\right] Q(t) H(t) q(t) \left| w\left(\frac{t+h(t)}{2}\right)\right|^m \operatorname{sgn} w\left(\frac{t+h(t)}{2}\right) = 0 \tag{15}$$

*is oscillatory, then equation (E) is oscillatory.*

Theorems 1 and 1′ and Corollary 1 are applicable to equations of type (E) only when $m = c + c^* \le 1$. Our next result provides sufficient conditions for the oscillation of (E) when $c \le 1$ and $c^* \le 1$.

THEOREM 2. *Let Condition (P) hold and $g(t) \le t$ and $g'(t) \ge 0$ for $t \ge t_0$. If for every positive constants $B_i$, $i = 1, 2$, the equations*

$$y'(t) + \left[\frac{B_1}{((n-1)!)^{c^*}}\right] (h'(t))^c (g(t))^{(n-1)c^*} q(t) \left|y(g(t))\right|^{c^*} \operatorname{sgn} y(g(t)) = 0 \tag{16}$$

*and*

$$z'(t) + \left[\frac{B_2}{(2^{n-2}(n-2)!)^c}\right] (t-h(t))^{(n-2)c} (h'(t))^c q(t) \left| z\left(\frac{t+h(t)}{2}\right)\right| \operatorname{sgn} z\left(\frac{t+h(t)}{2}\right) = 0 \tag{17}$$

*are oscillatory, then equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \ge t_0 \ge 0$. As in Theorem 1, we have Cases (I) and (II) for $t \ge t_1$.

Assume (I) holds. Then there exists a $t_2 \ge t_1$ and positive constants $b$ and $C$ such that

$$\frac{d}{dt} x(h(t)) \ge C h'(t), \qquad \text{for } t \ge t_2 \tag{18}$$

and

$$x(g(t)) \geq \frac{b}{(n-1)!}(g(t))^{n-1}x^{(n-1)}(g(t)), \qquad \text{for } t \geq t_2. \tag{19}$$

Using (P), (18), and (19) in (E), we get

$$w'(t) + \left[\frac{C^c b^{c^*}}{((n-1)!)^{c^*}}\right](h'(t))^c(g(t))^{(n-1)c^*}q(t)(w(g(t)))^{c^*} \leq 0, \qquad t \geq t_2,$$

where $w(t) = x^{(n-1)}(t)$, $t \geq t_2$. Now proceeding as in Theorem 1(I), we obtain the desired contradiction.

Assume (II) holds. Then there exist a $T \geq t_1$ and a positive constant $C_1$ such that (9) holds and

$$x(g(t)) \geq C_1, \qquad \text{for } t \geq T. \tag{20}$$

Thus, (10) takes the form

$$u'(t) + \left[\frac{C_1^{c^*}}{(2^{n-2}(n-2)!)^c}\right](t-h(t))^{(n-2)c}(h'(t))^c q(t)\left(u\left(\frac{t+h(t)}{2}\right)\right)^c \leq 0, \qquad t \geq T.$$

The rest of the proof is similar to that of Theorem 1(II). ∎

The following theorem provides sufficient conditions for the oscillation of (E) when $c$ and $c^*$ are arbitrary constants.

THEOREM 3. *Let condition (P) hold, $g(t) \leq t$ and $g'(t) \geq 0$ for $t \geq t_0$. If for every positive constants $B_i$, $i = 1, 2$, the equation*

$$y^{(n)}(t) + B_1\,(h'(t))^c\,q(t)\,|y(g(t))|^{c^*}\operatorname{sgn} y(g(t)) = 0 \tag{21}$$

*is oscillatory, and every bounded solution of the equation*

$$z^{(n-1)}(t) + B_2\,(h'(t))^c\,q(t)\,|z(h(t))|^c\operatorname{sgn} z(h(t)) = 0 \tag{22}$$

*is oscillatory, then equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. As in Theorem 1, Cases (I) and (II) for $t \geq t_1 \geq t_0$ hold.

In Case (I), (18) holds for $t \geq t_2 \geq t_1$. Thus, equation (E) leads to

$$x^{(n)}(t) + C^{c^*}\,(h'(t))^c\,q(t)(x(g(t)))^{c^*} \leq 0, \qquad \text{for } t \geq t_2.$$

However, then by a result of Foster and Grimmer [17], the equation

$$x^{(n)}(t) + C^{c^*}\,(h'(t))^c\,q(t)(x(g(t)))^{c^*} = 0$$

has a positive solution, which is a contradiction.

If (II) holds, then (20) is satisfied for $t \geq T \geq t_1$, and hence, we have

$$v^{(n-1)}(t) + (C_1 h'(t))^c\,q(t)(v(h(t)))^c \leq 0, \qquad \text{for } t \geq T, \tag{23}$$

where $v(t) = x'(t)$ and (8) holds for $t \geq T$. Integrating (23), $(n-1)$-times from $t$ to $T^* \geq T$, using (8) and letting $T^* \to \infty$, we have

$$v(t) \geq C_1^c \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!}\,(h'(s))^c\,q(s)(v(h(s)))^c\,ds.$$

The function $v(t)$ is positive and strictly decreasing on $[T, \infty)$. Thus, Theorem 1 in [15] ensures the existence of positive solution $z$ of equation (22) with $\lim_{t\to\infty} z(t) = 0$, a contradiction. This completes the proof. ∎

For the oscillatory behavior of all bounded solutions of (E), one can easily extract the following result.

COROLLARY 2. *Let condition (P) hold and $h(t) \le g(t) \le t$ for $t \ge t_0$. Moreover, assume that equation (2) is oscillatory for every constant $a_2$, $0 < a_2 < 1$, or every bounded solution of equation (22) is oscillatory for every $B_2 > 0$. Then all bounded solutions of (E) are oscillatory.*

The above results are not directly applicable to equations of type ($E_1$) when $n$ is even and the function $F$ satisfies condition (P) with $c = 0$. However, in this case also, we can employ the same technique to obtain the following result.

COROLLARY 3. *Let condition (P) hold with $c = 0$ and assume that there exists a nondecreasing continuous function $g^* : [t_0, \infty) \to (0, \infty)$ such that $g^*(t) \le \min\{t, g(t)\}$. If for every $b$, $0 < b < 1$, equation*

$$y'(t) + \left[\frac{b}{((n-1)!)^{c^*}}\right] (g^*(t))^{(n-1)c^*} |y(g^*(t))|^{c^*} \operatorname{sgn} y(g^*(t)) = 0 \tag{24}$$

*is oscillatory, then equation (E) (or ($E_1$) with $a = 1$) is oscillatory.*

The following example illustrates the importance of our results.

EXAMPLE 1. Consider the equation

$$x^{(n)}(t) + q(t) \left| x\left(\frac{t}{2}\right)\right|^{c^*} \left|\frac{d}{dt} x\left(\frac{t}{2}\right)\right|^{c} \operatorname{sgn} x\left(\frac{t}{2}\right) = 0, \qquad t \ge 1, \tag{25}$$

where $q : [1, \infty) \to (0, \infty)$ is continuous and $c$ and $c^*$ are positive constants. We consider the following cases.

(i) Let $c + c^* = m \le 1$. From Theorem 1, equation (25) is oscillatory if for every $a_1$ and $a_2 > 0$, the equations

$$y'(t) + \left[\frac{a_1}{(n-1)^{c^*}\left(2^{n-1}(n-2)!\right)^m}\right] t^{(n-2)m+c^*} q(t) \left|y\left(\frac{t}{2}\right)\right|^m \operatorname{sgn} y\left(\frac{t}{2}\right) = 0$$

and

$$z'(t) + \left[\frac{a_2}{\left(2^{2n-3}(n-2)!\right)^m}\right] t^{(n-2)m+c^*} q(t) \left|z\left(\frac{3t}{4}\right)\right|^m \operatorname{sgn} z\left(\frac{3t}{4}\right) = 0$$

are oscillatory. Also, (25) is oscillatory by Corollary 1, if we take $q(t) = t^{-(n-2)m-c^*-k}$, where $k$ is a constant, $0 < k < 1$ when $m = 1$ or $k = 1$ when $m < 1$.

(ii) Let $c \le 1$ and $c^* \le 1$. Then by Theorem 2, equation (25) is oscillatory if for every positive constants $B_1$ and $B_2$, the equations

$$y'(t) + \left[\frac{B_1}{2^c\left(2^{n-1}(n-1)!\right)^{c^*}}\right] t^{(n-1)c^*} q(t) \left|y\left(\frac{t}{2}\right)\right|^{c^*} \operatorname{sgn} y\left(\frac{t}{2}\right) = 0$$

and

$$z'(t) + \left[\frac{B_2}{\left(2^{2n-3}(n-2)!\right)^c}\right] t^{(n-2)c} q(t) \left|z\left(\frac{3t}{4}\right)\right|^c \operatorname{sgn} z\left(\frac{3t}{4}\right) = 0$$

are oscillatory. We also note that (25) is oscillatory if we take $q(t) = t^{n-2-k}$, $0 < k < 1$ when $c = c^* = 1$, and $q(t) = (1/t)\min\left\{t^{(n-1)c^*}, t^{(n-2)c}\right\}$, $t > 1$ when $c^* < 1$ and $c < 1$.

(iii) For any $c^* > 0$ and $c \le 1$, we can apply Theorem 3 and conclude that (25) is oscillatory if for every positive constants $d_1$ and $d_2$, the equations

$$y^{(n)}(t) + \left(\frac{d_1}{2^c}\right) q(t) \left|y\left(\frac{t}{2}\right)\right|^{c^*} \operatorname{sgn} y\left(\frac{t}{2}\right) = 0$$

and

$$z^{(n-1)}(t) + \left(\frac{d_2}{2^c}\right) q(t) \left|z\left(\frac{t}{2}\right)\right|^{c} \operatorname{sgn} z\left(\frac{t}{2}\right) = 0$$

are oscillatory. We can also select the function $q(t)$ depending on $c$ and $c^*$ and apply results of [3–6] to establish the oscillatory behavior of (25).

Finally, we remark that the oscillation results presented in [12] and other known oscillatory criteria in the literature are not applicable to equation (25).

## 4. OSCILLATION FOR ODD ORDER EQUATIONS

THEOREM 4. *Let condition (P) hold, $g'(t) \geq 0$ and $g(t) \geq t$ for $t \geq t_0$. If for every positive constants $a_i$, $i = 1, 2, 3$, the equations*

$$y'(t) + \left[\frac{a_1}{(2^{n-2}(n-2)!)^c}\right] \left(h'(t)(t-h(t))^{n-2}\right)^c q(t) \left|y\left(\frac{t+h(t)}{2}\right)\right|^c \operatorname{sgn} y\left(\frac{t+h(t)}{2}\right) = 0 \quad (26)$$

*and*

$$z'(t)+\left[\frac{a_2}{(2(n-2)!)^c}\right] \left(h'(t)(t-h(t))(h(t))^{n-2}\right)^c q(t) \left|z\left(\frac{t+h(t)}{2}\right)\right|^c \operatorname{sgn} z\left(\frac{t+h(t)}{2}\right) = 0 \quad (27)$$

*are oscillatory, and the differential inequality*

$$\begin{aligned} w'(t) - \left[\frac{a_3}{(2^{n-1}(n-1)!)^{c^*}}\right] \left(h'(t)(h(t))^{n-2}\right)^c (g(t)-t)^{(n-1)c^*} q(t) \left|w\left(\frac{t+g(t)}{2}\right)\right|^{c^*} \\ \times \operatorname{sgn} w\left(\frac{t+g(t)}{2}\right) \geq 0 \end{aligned} \quad (28)$$

*has no eventually positive solution, then equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 1, there exists a $t_1 \geq t_0$ such that $x'(t) > 0$ for $t \geq t_1$. We need to consider the following three cases:

(i) $x^{(n)}(t) \geq 0$, $x^{(n-1)}(t) < 0, x^{(n-2)}(t) > 0, \dots, x''(t) < 0$, $t \geq t_1$,
(ii) $x^{(n)}(t) \geq 0$, $x^{(n-1)}(t) < 0, \dots, x''(t) > 0$, $t \geq t_1$,
(iii) $x^{(i)}(t) > 0$, $0 \leq i \leq n$ and $t \geq t_1$.

Assume (i) holds. Since $x(t)$ is an increasing function for $t \geq t_1$, there exist a $t_2 \geq t_1$ and a positive constant $k$ such that

$$x(g(t)) \geq k, \qquad \text{for } t \geq t_2. \quad (29)$$

Using condition (P) and (29) in (E), we get

$$x^{(n)}(t) \geq k^{c^*} q(t) \left(\frac{d}{dt} x(h(t))\right)^c, \qquad \text{for } t \geq t_2. \quad (30)$$

Setting $v(t) = x'(t), t \geq t_2$ to obtain

$$v^{(n-1)}(t) \geq k^{c^*} q(t)(h'(t))^c (v(h(t)))^c, \qquad \text{for } t \geq t_2. \quad (31)$$

By Lemma 1 in [8], there exists a $t_3 \geq t_2$ such that

$$\begin{aligned} v(h(t)) &\geq \frac{(t-h(t))^{n-2}}{2^{n-2}(n-2)!} \left(-v^{(n-2)}\left(\frac{t+h(t)}{2}\right)\right) \\ &= \frac{(t-h(t))^{n-2}}{2^{n-2}(n-2)!} z\left(\frac{t+h(t)}{2}\right), \qquad \text{for } t \geq t_3, \end{aligned} \quad (32)$$

where $z(t) = -v^{(n-2)}(t) > 0$ for $t \geq t_1$. Thus, inequality (31) can be written as

$$z'(t) + \left[\frac{k^{c^*}}{(2^{n-2}(n-2)!)^c}\right] \left((t-h(t))^{n-2} h'(t)\right)^c q(t) \left(z\left(\frac{t+h(t)}{2}\right)\right)^c \leq 0, \qquad \text{for } t \geq t_3.$$

Now proceeding as in Theorem 1(I), we find the desired contradiction.

Assume (ii) holds. As in (i), we obtain (29) for $t \geq t_2$. By Lemma 2, there exist a $T_1 \geq t_2$ and a positive constant $b$, $0 < b < 1$ such that

$$\frac{d}{dt}x(h(t)) = x'(h(t))h'(t) \geq \left(\frac{b}{(n-2)!}\right), \qquad (h(t))^{n-2}h'(t)x^{(n-2)}(h(t)), \qquad t \geq T_1.$$

Thus,

$$u''(t) \geq \left[\frac{k^{c^*}b^c}{((n-2)!)^c}\right]\left((h(t))^{n-2}h'(t)\right)^c q(t)(u(h(t)))^c, \qquad t \geq T_1, \tag{33}$$

where $u(t) = x^{(n-2)}(t) > 0$ for $t \geq T_1$. Next, by Lemma 1 in [8], there exists a $T_2 \geq T_1$ such that

$$u(h(t)) \geq \frac{(t-h(t))}{2}\left(-u'\left(\frac{t+h(t)}{2}\right)\right), \qquad \text{for } t \geq T_2,$$

and consequently, (33) leads to

$$w'(t) + \left[\frac{k^{c^*}b^c}{(2(n-2)!)^c}\right]\left((h(t))^{n-2}h'(t)(t-h(t))\right)^c q(t)\left(w\left(\frac{t+h(t)}{2}\right)\right)^c \leq 0, \qquad t \geq T_2,$$

where $w(t) = -u'(t) > 0$ for $t \geq T_2$. The rest of the proof is similar to that of Theorem 1(I).

Finally, assume (iii) holds. Then, there exist a $T^* \geq t_1$ and a positive constant $B$ such that

$$\frac{d}{dt}x(h(t)) \geq B(h(t))^{n-2}h'(t), \qquad \text{for } t \geq T^*,$$

and hence, equation (E) gives

$$x^{(n)}(t) \geq B^c\left((h(t))^{n-2}h'(t)\right)^c q(t)(x(g(t)))^{c^*}, \qquad \text{for } t \geq T^*. \tag{34}$$

Let $T \geq T^*$. From Taylor's formula

$$x(u) = \sum_{i=0}^{n-1}\frac{(u-v)^i}{i!}x^{(i)}(v) + \frac{1}{(n-1)!}\int_v^u (u-t)^{n-1}x^{(n)}(t)\,dt,$$

Case (iii) leads to

$$x(u) \geq \frac{(u-v)^{n-1}}{(n-1)!}x^{(n-1)}(v) \text{ for } u \geq v \geq T.$$

Putting $u = g(t)$ and $v = (t+g(t))/2$ in the above inequality, we get

$$x(g(t)) \geq \frac{(g(t)-t)^{n-1}}{2^{n-1}(n-1)!}x^{(n-1)}\left(\frac{t+g(t)}{2}\right), \qquad g(t) \geq \frac{t+g(t)}{2} \geq T. \tag{35}$$

Using (35) in (34), we obtain

$$W'(t) \geq \left[\frac{B^c}{(2^{n-1}(n-1)!)^{c^*}}\right]\left((h(t))^{n-2}h'(t)\right)^c (g(t)-t)^{(n-1)c^*} q(t)\left(W\left(\frac{t+g(t)}{2}\right)\right)^{c^*}, \qquad \text{for } t \geq T,$$

where $W(t) = x^{(n-1)}(t)$ is a positive solution for $t \geq T$ of the above inequality, a contradiction to our assumption. This completes the proof. ∎

If we let $a = \min\{a_1, a_2\}$ and

$$Q(t) = \min\left\{\left(\frac{h'(t)(t-h(t))^{n-2}}{2^{n-2}}\right)^c, \left(\frac{h'(t)(t-h(t))(h(t))^{n-2}}{2}\right)^2\right\},$$

then Theorem 4 can be restated as follows.

THEOREM 4'. *Let condition (P) hold, $g'(t) \geq 0$ and $g(t) \geq t$ for $t \geq t_0$. If for every positive constants $a$ and $a_3$, the equation*

$$y'(t) + \left[\frac{aQ(t)}{((n-2)!)^c}\right] q(t) \left| y\left(\frac{t+h(t)}{2}\right)\right|^c \operatorname{sgn} y\left(\frac{t+h(t)}{2}\right) = 0 \tag{36}$$

*is oscillatory, and the inequality (28) has no eventually positive solution, then equation (E) is oscillatory.*

THEOREM 5. *Let condition (P) hold, $g'(t) \geq 0$ and $g(t) \geq t$ for $t \geq t_0$. If for all positive constants $a_1$ and $a_3$, equation (26) is oscillatory and the inequality (28) has no eventually positive solution, and*

$$\int^{\infty} s\,(h'(s))^c q(s)\,ds = \infty, \tag{37}$$

*then equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. We proceed as in Theorem 4 and only consider Case (ii). Since $x''(t) > 0$, $x'(t) > 0$ for $t \geq t_1$, there exist a $T \geq t_1$ and positive constants $d_1$ and $d_2$ such that

$$x(g(t)) \geq d_1 \quad \text{and} \quad \frac{d}{dt}x(h(t)) \geq d_2 h'(t), \qquad \text{for } t \geq T. \tag{38}$$

We multiply equation (E) by $t$, use condition (P) and (38) and integrate from $T$ to $t \geq T$, to obtain

$$tx^{(n-1)}(t) - \int_T^t x^{(n-1)}(s)\,ds \geq N + d_1^{c^*} d_2^c \int_T^t s\,(h'(s))^c q(s)\,ds,$$

where $N$ is a real number. Now by (37), we get

$$\lim_{t\to\infty} \left[tx^{(n-1)}(t) - x^{(n-2)}(t)\right] = \infty,$$

and thus, by Lemma 1 in [7], we find that $x^{(n-1)}(t) \to \infty$ as $t \to \infty$, a contradiction. This completes the proof. ∎

THEOREM 6. *Let condition (P) hold, $g'(t) \geq 0$ and $g(t) \geq t$ for $t \geq t_0$. If for every positive constants $a_1$, $a_2$, and $a_3$, the bounded solutions of the equations*

$$y^{(n-1)}(t) - a_1 (h'(t))^c q(t) |y(h(t))|^c \operatorname{sgn} y(h(t)) = 0, \tag{39}$$

$$z''(t) - \left[\frac{a_2}{((n-2)!)^c}\right] \left((h(t))^{n-2} h'(s)\right)^c q(t) |z(h(t))|^c \operatorname{sgn} z(h(t)) = 0 \tag{40}$$

*are oscillatory and the inequality (28) has no eventually positive solution, then (E) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. As in Theorem 4, we have Cases (i)–(iii). The proof of Case (iii) is similar to that of Theorem 4(iii). Now we assume (i) holds. As in Theorem 4(i), we see that (29) holds for $t \geq t_2$. Integrating (31), $(n-1)$ times from $t$ to $T \geq t \geq t_2$ and using the fact that $(-1)^i v^{(i)}(t) > 0$ for $t \geq t_2$ and $0 \leq i \leq n-1$, we have

$$v(t) \geq k^{c^*} \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} (h'(s))^c q(s) (v(h(s)))^c\,ds, \qquad t \geq t_2.$$

We proceed as in Theorem 3(II) and obtain the desired contradiction. Finally, we assume that (ii) holds. As in the proof of Theorem 4(ii), we note that (33) holds for $t \geq T_1$. Once again, we integrate (33) from $t$ to $t^* \geq T_1$ and let $T^* \to \infty$, to get

$$u(t) \geq \left[\frac{k^{c^*} b^c}{((n-2)!)^c}\right] \int_t^{\infty} (s-t) \left((h(s))^{n-2} h'(s)\right)^c q(s) (u(h(s)))^c\,ds, \qquad t \geq T_1.$$

The rest of the proof is similar to that of Theorem 3(II). ∎

Next, we give some applications of our comparison theorems in order to illustrate their validity and demonstrate their significance. More precisely, we apply the results of [14] and their analogs to advanced equations, and other results for first-order equations and/or inequalities appeared in [6], and state the following new criterion.

COROLLARY 4. *Let Condition (P) hold, $g'(t) \geq 0$ and $g(t) \geq t$ for $t \geq t_0$. If for all positive constants $a_1, a_2$ and $a_3$,*

$$\liminf_{t\to\infty} \int_{(t+h(t))/2}^{t} h'(s)(s-h(s))^{n-2} q(s)\,ds > \frac{2^{n-2}(n-2)!}{a_1 e}, \qquad \text{when } c = 1 \tag{41}$$

*or*

$$\int^{\infty} \left(h'(s)(s-h(s))^{n-2}\right)^c q(s)\,ds = \infty, \qquad \text{when } c < 1, \tag{42}$$

$$\liminf_{t\to\infty} \int_{(t+h(t))/2}^{t} h'(s)(s-h(s))(h(s))^{n-2} q(s)\,ds > \frac{2(n-2)!}{a_2 e}, \qquad \text{when } c = 1 \tag{43}$$

*or*

$$\int^{\infty} \left(h'(s)(s-h(s))(h(s))^{n-2}\right)^c q(s)\,ds = \infty, \qquad \text{when } c < 1, \tag{44}$$

*and either*

$$\liminf_{t\to\infty} \int_{t}^{(t+g(t))/2} \left(h'(s)(h(s))^{n-2}\right)^c (g(s)-s)^{n-1} q(s)\,ds > \frac{2^{n-1}(n-1)!}{a_3 e}, \quad \text{when } c^* = 1 \text{ and } c > 0 \tag{45}$$

*or*

$$\int^{\infty} \left(h'(s)(h(s))^{n-2}\right)^c (g(s)-s)^{(n-1)c^*}\,ds = \infty, \qquad \text{when } c^* > 1 \text{ and } c > 0, \tag{46}$$

*then (E) is oscillatory.*

The following result establishes the oscillation of all bounded solutions of equation (E).

COROLLARY 5. *Let condition (P) hold and assume that equation (26) is oscillatory for every constant $a_1 > 0$, or every bounded solution of (39) is oscillatory for every constant $a_1 > 0$. Then every bounded solution of equation (E) is oscillatory.*

PROOF. Let $x(t)$ be a bounded and nonoscillatory solution of (E). Assume $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 1, there exists a $t_1 \geq t_0$ such that (i) holds for $t \geq t_1$. Proceeding as in Theorem 4(i) or Theorem 6(i), we obtain the desired contradiction. ∎

We note that the inequality (28) in Theorems 4–6 can be replaced by an equation of the same form. However, in such a case we need an analogous result similar to that of Theorem 1 in [17] for advanced type equations. We also note that the results of Section 4 are not directly applicable to equations of type ($E_1$) when $n$ is odd, $a = -1$ and the function $F$ satisfies condition (P) with $c = 0$. However, we can make use of techniques of the proofs employed here and obtain new criteria similar to those of Theorems 4–6. Finally, we illustrate the following example.

EXAMPLE 2. Consider equation

$$x^{(n)}(t) - t^{2-n} \left| x\left(\frac{3t}{2}\right)\right|^{c^*} \left|\frac{d}{dt} x\left(\frac{t}{2}\right)\right|^{c} \operatorname{sgn} x\left(\frac{3t}{2}\right) = 0, \qquad t \geq 1, \tag{47}$$

where $c$ and $c^*$ are positive constants. All conditions of Corollary 4 are satisfied for $0 < c \leq 1$ and $c^* \geq 1$, and hence, we conclude that (47) is oscillatory. It is interesting to note that the known oscillatory criteria are not applicable to equation (47).

## REFERENCES


1. R.P. Agarwal, Oscillation and asymptotic behavior of solutions of differential equations with nested arguments, *Bulletin UMI, Serie IV,* Special Issue on Analisi Funzionale e Applicazioni **1–C**, 137–146, (1982).
2. R.P. Agarwal and E. Thandapani, Asymptotic behavior and oscillation of solutions of differential equations with deviating arguments, *Bulletin UMI* **17–B**, 82–93, (1980).

3. S.R. Grace and B.S. Lalli, Oscillation theorems for $n^{\text{th}}$ order delay differential equations, *J. Math. Anal. Appl.* **91**, 352–366, (1983).
4. A.G. Kartasatos, *Lecture Notes in Pure and Appl. Math.*, Vol. 28, pp. 17–72, Marcel Dekker, New York, (1977).
5. I.T. Kiguradze, On the oscillation of solutions of equation $\frac{d^m u}{dt^m} + a(t)|u|^m \operatorname{sgn} u = 0$, *Mat. Sb.* (in Russian) **65**, 172–187, (1964).
6. Y. Kitamura, Oscillation of functional differential equations with general deviating arguments, *Hiroshima Math. J.* **15**, 445–491, (1985).
7. V.A. Staikos and Y.G. Sficas, Forced oscillations for differential equations of arbitrary order, *J. Differential Equations* **17**, 1–11, (1975).
8. V.A. Staikos and I.P. Stavroulakis, Bounded oscillations under the effect of retardations for differential equations of arbitrary order, *Proc. Roy. Soc. Edinburgh* **77A**, 129–136, (1977).
9. S.R. Grace, Oscillatory and asymptotic behavior of certain functional differential equations, *J. Math. Anal. Appl.* **162**, 177–188, (1991).
10. S.R. Grace, Oscillation theorems for damped functional differential equations, *Funkcialaj Ekvac.* **35**, 261–278, (1992).
11. S.R. Grace, Oscillation theorems for certain functional differential equations, *J. Math. Anal. Appl.* **184**, 100–111, (1994).
12. S.R. Grace, Oscillatory and asymptotic behavior of delay differential equations with a nonlinear damping term, *J. Math. Anal. Appl.* **168**, 306–318, (1992).
13. Ch.G. Philos, A new criterion for the oscillatory and asymptotic behavior of delay differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Mat.* **39**, 61–64, (1981).
14. R.G. Kopladze and T.A. Chanturija, On oscillatory and monotone solutions of first order differential equations with deviating arguments, *Differential'nye Uravenniji* (in Russian) **18**, 1463–1465, (1982).
15. G. Ladas and I.P. Stavroulakis, Oscillations caused by several retarded and advanced arguments, *J. Differential Equations* **44**, 134–152, (1982).
16. K.E. Foster and R.C. Grimmer, Nonoscillatory solutions of higher order delay equations, *J. Math. Anal. Appl.* **77**, 150–164, (1980).
17. Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at $\infty$ for differential equations with positive delays, *Arch. Math.* **36**, 168–178, (1981).
18. J.S.W. Wong and T.A. Burton, Some properties of $u'' + a(t)f(u)g(u') = 0$, *Monatsh Math.* **69**, 364–374, (1965).